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Reduction of Ordering Effect in Reliability-Based Design Optimization Using Dimension Reduction Method

Yoojeong Noh,* K. K. Choi,† and Ikjin Lee‡
University of Iowa, Iowa City, Iowa 52242

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In reliability-based design optimization problems with correlated input variables, a joint cumulative distribution function needs to be used to transform the correlated input variables into independent standard Gaussian variables for the inverse reliability analysis. To obtain a true joint cumulative distribution function, a very large number of data (if not infinite) needs to be used, which is impractical in industry applications. In this paper, a copula is proposed to model the joint cumulative distribution function using marginal cumulative distribution functions and correlation parameters obtained from samples. Using the joint cumulative distribution function modeled by the copula, the transformation and the first-order reliability method can be carried out. However, the first-order reliability method may yield different reliability analysis results for different transformation ordering of input variables. Thus, the most probable-point-based dimension reduction method, which is more accurate than the first-order reliability method and more efficient than the second-order reliability method, is proposed for the inverse reliability analysis to reduce the effect of transformation ordering.

Nomenclature

$C(\cdot\|\boldsymbol{\theta})$	=	copula with $\boldsymbol{\theta}$
$\mathbf{d}$	=	vector of design variables, $[d_1, \ldots, d_{\mathrm{ndv}}]^T$
$\mathbf{E}$	=	vector of correlated standard elliptical variables, $[E_1, \ldots, E_n]^T$
$\boldsymbol{e}$	=	realization of vector $\mathbf{E}$, $[e_1, \ldots, e_n]^T$
$F_{X_i}(\cdot)$	=	marginal cumulative distribution function of X_i
$F_{X_i}(\cdot\|\cdot)$	=	conditional cumulative distribution function of X_i
$F_{X_1,\ldots,X_n}(\cdot)$	=	joint cumulative distribution function of $X_1, \ldots, X_n$
$f_{X_1,\ldots,X_n}(\cdot)$	=	joint probability density function of $X_1, \ldots, X_n$
n	=	number of random variables
$\mathbf{P}$	=	covariance matrix of $\mathbf{X}$, $\{\rho_{ij}\}$
$\mathbf{P}'$	=	covariance matrix of $\mathbf{E}$, $\{\rho'_{ij}\}$
$\mathbf{U}$	=	vector of independent standard Gaussian variables, $[U_1, \ldots, U_n]^T$
$\mathbf{X}$	=	vector of random variables, $[X_1, \ldots, X_n]^T$
$\mathbf{x}$	=	realization of vector $\mathbf{X}$, $[x_1, \ldots, x_n]^T$
$\mathbf{x}^*$	=	most probable point based on the first-order reliability method
$\mathbf{x}^*_{\mathrm{DRM}}$	=	most probable point based on the dimension reduction method
$\boldsymbol{\theta}$	=	matrix of correlation parameters of $X_1, \ldots, X_n$
ρ_{ij}	=	Pearson's correlation coefficient between X_i and X_j
τ	=	Kendall's tau
$\Phi(\cdot)$	=	marginal Gaussian cumulative distribution function
$\Phi_{\mathbf{P}'}(\cdot\|\mathbf{P}')$	=	multivariate Gaussian cumulative distribution function with $\mathbf{P}'$

I. Introduction

IN MANY reliability-based design optimization (RBDO) problems, input random variables such as the material properties and fatigue parameters are correlated [1–3]. For the RBDO problem with the correlated input variables, the joint cumulative distribution function (CDF) of the input variables should be available to transform the correlated input variables into the independent standard Gaussian variables by using the Rosenblatt transformation [4] to carry out the inverse reliability analysis. However, in industrial applications, often only the marginal CDFs and limited paired sampled data are available using experimental testing, and the input joint CDF is very difficult to obtain. In this paper, a copula, which links the joint CDF and marginal CDFs, is used to model the joint CDF. Because the copula only requires marginal CDFs and correlation parameters, which are often available in industrial applications, the joint CDF can be readily obtained. Thus, it is desirable to use the copula for modeling the joint CDFs in practical applications with correlated input variables.

Once the joint CDF is obtained using the copula, the Rosenblatt transformation can be used to transform the original random variables into the independent standard Gaussian variables for the inverse reliability analysis. For the inverse reliability analysis, the first-order reliability method (FORM) is most often used. On the other hand, depending on the types of the joint input CDFs, if a different order of Rosenblatt transformation is used, even the constraint function that was mildly nonlinear with respect to the original random variables could become highly nonlinear in terms of the independent standard Gaussian variables. First, obviously, if the input variables are independent (i.e., the joint CDF is a simple multiplication of the marginal CDFs), there is no effect of transformation ordering. Second, if the input variables have the joint CDF modeled by an elliptical copula, the effect of transformation ordering still does not exist, because the elliptical copula makes the Rosenblatt transformation become linear, which is independent of orderings. However, if the input variables have a joint CDF modeled by a nonelliptical copula, which often occurs in industrial applications [5], because the Rosenblatt transformation becomes highly nonlinear, the different ordering can significantly affect the nonlinearity of the transformed constraints. In this case, if the FORM is used, the inverse reliability analysis results could be very different for the different ordering, because the FORM uses a linear

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*Graduate Research Assistant, Department of Mechanical and Industrial Engineering, College of Engineering; noh@engineering.uiowa.edu.

†Roy J. Carver Professor, Department of Mechanical and Industrial Engineering, College of Engineering; kkchoi@engineering.uiowa.edu (Corresponding Author).

‡Postdoctoral Research Scholar, Department of Mechanical and Industrial Engineering, College of Engineering; ilee@engineering.uiowa.edu.

approximation of the constraint to estimate the probability of failure. This effect of transformation ordering in RBDO will be unacceptable and will make the user significantly concerned.

To reduce the dependency of the inverse reliability analysis result and thus the RBDO result on the ordering of the Rosenblatt transformation, it is proposed to use the most probable point (MPP)-based dimension reduction method (DRM) [6] for the inverse reliability analysis in this paper. With the accuracy of the inverse reliability analysis using the DRM even for highly nonlinear constraint functions, it is shown that the RBDO results are becoming less dependent on the Rosenblatt transformation ordering of the input variables.

II. Modeling of a Joint CDF Using a Copula

As mentioned earlier, if the input variables are correlated, it is often too difficult to obtain the true joint CDF in practical industrial applications with only limited experimental data. In this paper, a copula is used to model the joint CDF using marginal CDFs and correlation measures that are calculated from the experimental data. The definition of copula and the correlation measures associated with copulas are explained in this section.

A. Definition of Copula

The word *copula* originated from a Latin word for “link” or “tie” that connects two different things. In statistics, the definition of copula is stated in [7]: “Copulas are functions that join or couple multivariate distribution functions to their one-dimensional marginal distribution functions. Alternatively, copulas are multivariate distribution functions for which the one-dimensional margins are uniform on the interval [0, 1].”

According to Sklar’s theorem, if the random variables have a joint distribution $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$ with marginal distributions $F_{X_1}(x_1),\ldots,F_{X_n}(x_n)$, then there exists an n-dimensional copula C such that

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = C(F_{X_1}(x_1),\ldots,F_{X_n}(x_n)|\boldsymbol{\theta}) \tag{1}$$

where $\boldsymbol{\theta}$ is the matrix of the correlation parameters of $x_1,\ldots,x_n$. If marginal distributions are all continuous, then C is unique. Conversely, if C is an n-dimensional copula and $F_{X_1}(x_1),\ldots,F_{X_n}(x_n)$ are the marginal distributions, then $F_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$ is the joint distribution [7]. By taking the derivative of Eq. (1), the joint probability density function (PDF) $f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$ is obtained as

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = c(F_{X_1}(x_1),\ldots,F_{X_n}(x_n)|\boldsymbol{\theta})\prod_{i=1}^{n} f_{X_i}(x_i) \tag{2}$$

where

$$c(z_1,\ldots,z_n) \equiv \frac{\partial^n C(z_1,\ldots,z_n)}{\partial z_1,\ldots,\partial z_n}$$

with $z_i = F_{X_i}(x_i)$, and $f_{X_i}(x_i)$ is the marginal PDF for $i = 1,\ldots,n$.

A copula only requires marginal CDFs and correlation parameters to model a joint CDF, and so the joint CDF can be readily obtained from limited data. In addition, because the copula decouples marginal CDFs from the joint CDF, the joint CDF modeled by the copula can be expressed in terms of any type of marginal CDF. That is, having marginal Gaussian CDFs does not mean that the joint CDF is Gaussian. Thus, it is desirable to be able to model the joint CDF of correlated input variables with mixed types of marginal CDFs, which can often occur in industrial applications [5]. To model the joint CDF using the copula, the correlation parameters need to be obtained from experimental data, as seen in Eqs. (1) and (2). Because various types of copulas have their own correlation parameters, it is desirable to have a common correlation measure to obtain the correlation parameters from the experimental data.

B. Correlation Measures

To measure the correlation between two random variables, Pearson’s rho and Kendall’s tau can be used. Pearson’s rho was first discovered by Bravais [8] in 1846 and was developed by Pearson [9] in 1896. Pearson’s rho indicates the degree of linear relationship between two random variables as

$$\rho_{XY} = \frac{\mathrm{cov}(X,Y)}{\sigma_X\sigma_Y} \tag{3}$$

where σ_X and σ_Y are standard deviations of X and Y, respectively, and $\mathrm{cov}(X,Y)$ is the covariance between X and Y. Because Pearson’s rho only indicates the linear relationship between two random variables, it is valid only when the joint CDF is elliptical.

A random variable $\mathbf{X}$ has an elliptical distribution if and only if there exist $\mathbf{S}$, R, and $\mathbf{A}$ that satisfy

$$\mathbf{X} = \boldsymbol{\mu} + R\mathbf{AS} = \boldsymbol{\mu} + \mathbf{AE} \tag{4}$$

where R is a nonnegative random variable, which is independent of k-dimensional uniformly distributed random vector $\mathbf{S}$; $\mathbf{A}$ is a $d\times k$ matrix with $\mathbf{AA}^T = \mathbf{P}$, where $\mathbf{P}$ is the covariance matrix of $\mathbf{X}$ consisting of Pearsons’s rhos; and d and k are the number of random variables in $\mathbf{X}$ and $\mathbf{S}$, respectively [10,11]. It is only considered that rank $(\mathbf{A}) = d \le k$ and $\mathbf{P}$ is positive definite with full rank. When R^2 has a chi-square distribution with d degrees of freedom, $\mathbf{E}$ has a standard Gaussian distribution $N(\mathbf{0},\mathbf{1})$ and, accordingly, $\mathbf{X}$ has a Gaussian distribution $N(\boldsymbol{\mu},\mathbf{P})$ for d-dimensional variables. If R^2/d has F distribution with d and ν degrees of freedom, $\mathbf{E}$ has a standard t distribution $T_\nu(\nu,\mathbf{0},\mathbf{1})$ and $\mathbf{X}$ has a t distribution with $T_{\mathbf{P},\nu}(\nu,\boldsymbol{\mu},\mathbf{P})$ for d-dimensional variables. The t distribution and Gaussian distribution originate the t copula and the Gaussian copula, respectively [5].

The Gaussian copula is defined as

$$C_\Phi(z_1,\ldots,z_n|\mathbf{P}') = \Phi_{\mathbf{P}'}(\Phi^{-1}(z_1),\ldots,\Phi^{-1}(z_n)|\mathbf{P}'), \qquad \mathbf{z}\in I^n \tag{5}$$

where $z_i = F_{X_i}(x_i)$ is the marginal CDF of X_i for $i = 1,\ldots,n$, $\mathbf{P}' = \{\rho'_{ij}\}$ is the covariance matrix consisting of Pearson’s rho between correlated standard Gaussian variables $\Phi^{-1}(z_i)$ and $\Phi^{-1}(z_j)$, $\Phi(\cdot)$ represents the marginal standard Gaussian CDF,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$$

and $\Phi_{\mathbf{P}'}(\cdot)$ is the joint Gaussian CDF, defined as

$$\Phi_{\mathbf{P}'}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}}\exp\left[-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T(\mathbf{P}')^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$$

for $\mathbf{x} = [x_1,\ldots,x_n]^T$ with a mean vector $\boldsymbol{\mu} = [\mu_1,\ldots,\mu_n]^T$. Another popular elliptical distribution is a multivariate t distribution of standard t variables with $T_{\mathbf{P},\nu}(\mathbf{0},\mathbf{P}',\nu)$:

$$C_T(z_1,\ldots,z_n|\mathbf{P}',\nu) = T_{\mathbf{P}',\nu}(T_\nu^{-1}(z_1),\ldots,T_\nu^{-1}(z_n)|\mathbf{P}') \tag{6}$$

where ν is the degree of freedom, and $\mathbf{P}' = \{\rho'_{ij}\}$ is the covariance matrix between $T_\nu^{-1}(z_i)$ and $T_\nu^{-1}(z_j)$. In Eq. (6), $T_\nu^{-1}(\cdot)$ is the inverse of the Student t distribution with ν degree of freedom defined as

$$T_\nu(x) = \frac{1}{2} + \left\{\left[x\Gamma\left(\frac{1}{2}(x+1)\right){}_2F_1\left(\frac{1}{2},\frac{x+1}{2};\frac{3}{2};-\frac{x^2}{\nu}\right)\right]\Big/\sqrt{\pi x}\,\Gamma\left(\frac{x}{2}\right)\right\} \tag{7}$$

where $\Gamma(\cdot)$ and ${}_2F_1$ are gamma and hyper geometric functions, respectively. The joint t distribution is given by

$$T_{\mathbf{P}',\nu}(\mathbf{x}) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} \frac{\Gamma((\nu+n)/2)}{\Gamma(\nu/2)\sqrt{\nu^n \pi^n |\mathbf{P}'|}} \times \left\{1 + \frac{(\boldsymbol{\xi}-\boldsymbol{\mu})^T \mathbf{P}'^{-1}(\boldsymbol{\xi}-\boldsymbol{\mu})}{\nu}\right\}^{-\frac{n+\nu}{2}} \mathrm{d}\boldsymbol{\xi} \tag{8}$$

However, Pearson's rho, which is used as a correlation measure for the elliptical copulas, cannot be a good measure for a nonlinear relationship between two random variables. If the given data follow a joint CDF modeled by a nonelliptical copula, another correlation measure is necessary.

Unlike Pearson's rho, Kendall's tau does not require the assumption that the relationship between two random variables is linear. Because Kendall's tau measures the correspondence of rankings between correlated random variable, it is called a rank correlation coefficient. Kendall's tau was first introduced by Kendall [12] in 1938 and is defined as

$$\tau = 4 \iint_{I^2} C(z_1, z_2|\theta)\mathrm{d}C(z_1, z_2) - 1 \tag{9}$$

where $I^2 = I \times I (I = [0, 1])$ and Eq. (9) is the population version of Kendall's tau. The sample version of Kendall's tau is

$$t = \frac{c-d}{c+d} = \frac{2(c-d)}{\mathrm{ns}(\mathrm{ns}-1)}\beta \tag{10}$$

where c represents the number of concordant pairs, d is the number of discordant pairs, and ns is the number of samples. Using the estimated Kendall's tau, the correlation parameter of the copula, θ, can be calculated because Kendall's tau can be expressed as a function of the correlation parameter, as shown in Eq. (9). The explicit functions of Eq. (9) for some copulas are presented in [13].

Consider a nonelliptical copula, which uses a rank correlation coefficient such as Kendall's tau as the correlation measures. Unlike the elliptical copula, the Archimedean copula is constructed in a completely different way. An important component of constructing Archimedean copula is a generator function φ_θ with a correlation parameter θ. If φ_θ is a continuous and strictly decreasing function from [0,1] to $[0, \infty)$ such that $\varphi_\theta(0) = \infty$ and $\varphi_\theta(1) = 0$ and the inverse φ_θ^{-1} is completely monotonic on $[0, \infty)$, then the Archimedean copula can be defined as [7]

$$C(z_1, \ldots, z_n|\theta) = \varphi_\theta^{-1}[\varphi_\theta(z_1) + \cdots + \varphi_\theta(z_n)] \tag{11}$$

for $n \geq 2$, and $z_i = F_{X_i}(x_i)$. Each Archimedean copula has a corresponding unique generator function φ_θ, which provides a multivariate copula, as shown in Eq. (11). Once the generator function is provided, Kendall's tau can be obtained as

$$\tau = 1 + 4\int_0^1 \frac{\varphi_\theta(t)}{\varphi'_\theta(t)}\mathrm{d}t \tag{12}$$

where $\varphi'_\theta(t)$ is the derivative of the characteristic function $\varphi_\theta(t)$ with respect to t. Using Eq. (12), the correlation parameter θ can be expressed in terms of Kendall's tau.

The Archimedean copula can be used for modeling a multivariate CDF. But it is hard to expand to an n-dimensional copula, because, as shown in Eq. (11), it has one generator function and thus has the same correlation parameter, even if n variables are correlated with different correlation coefficients. Hence, most copula applications consider bivariate data. For multivariate data, the data are analyzed pair by pair using a bivariate copula. This paper also considers a bivariate copula. More detailed information on Kendall's tau is presented in [12].

Including the elliptical copula and Archimedean copula, there exist various kinds of copulas. Thus, selecting an appropriate copula is necessary to correctly model a joint CDF based on the given experimental data. As mentioned earlier, to model a joint CDF using a copula, the marginal CDFs and correlation parameters need to be obtained. The marginal CDFs are often known to follow specific CDF types; for example, some material properties such as fatigue parameters are known to follow lognormal CDFs. On the other hand, selecting an appropriate copula that best describes the given experimental data is not a simple problem. Because addressing two issues (effect of transformation ordering and identification of the right copula) together is complicated and requires lengthy discussion, in this paper, only the effect of transformation ordering is addressed, and the joint CDFs modeled by copulas are assumed to be exact. The identification of the right copula is addressed in [13,14] in detail.

III. Effect of Transformation Ordering in RBDO

Based on the identified joint CDF, the input variables need to be transformed into independent standard Gaussian variables for the inverse reliability analysis in RBDO using the Rosenblatt transformation. When the input variables are independent or the joint CDF is modeled by an elliptical copula, the ordering of input variables does not affect the transformation. However, when the joint CDF is modeled by a nonelliptical copula, different orderings of input variables cause different transformations for the inverse reliability analysis, which leads to different RBDO results. This issue will be addressed in this section.

A. Rosenblatt Transformation for RBDO

The RBDO problem can be formulated to

$$\text{minimize cost}(\mathbf{d}) \qquad \text{subject to } P(G_i(\mathbf{X}) > 0) \leq P_{F_i}^{\text{tar}}$$
$$i = 1, \ldots, \mathrm{nc}, \qquad d_L \leq \mathbf{d} \leq d_U, \qquad \mathbf{d} \in R^{\mathrm{ndv}}, \qquad \mathbf{X} \in R^n \tag{13}$$

where $\mathbf{X}$ is the vector of random variables; $\mathbf{d}$ is the vector of design variables, which is the mean value of the random variables $\mathbf{X}$, $\mathbf{d} = \mu(\mathbf{X})$; $G_i(\mathbf{X})$ represents the constraint functions; $P_{F_i}^{\text{tar}}$ is the given target probability of failure for the ith constraint; and nc, ndv, and n are the number of probabilistic constraints, number of design variables, and number of random variables, respectively.

The probability of failure is estimated by a multidimensional integral of the joint PDF of the input variables over the failure region as

$$P(G_i(\mathbf{X}) > 0) = \int_{G_i(\mathbf{X})>0} f_{\mathbf{X}}(\mathbf{x})\mathrm{d}\mathbf{x}, \qquad i = 1, \ldots, \mathrm{nc} \tag{14}$$

where $\mathbf{x}$ is the realization of the random vector $\mathbf{X}$. However, because it is difficult to compute the multidimensional integral analytically, approximation methods such as the FORM or the second-order reliability method (SORM) are used. The FORM often provides adequate accuracy and is much easier to use than the SORM, and hence it is commonly used in RBDO. Because the FORM and SORM require the transformation of the correlated random input variables into the standard Gaussian variables, the Rosenblatt transformation is used.

Using a performance measure approach (PMA) [15], the ith constraint can be rewritten from Eq. (13) as

$$P(G_i(\mathbf{X}) > 0) - P_{F_i}^{\text{tar}} \leq 0 \Rightarrow G_i(\mathbf{x}^*) \leq 0 \tag{15}$$

where $G_i(\mathbf{x}^*)$ is the ith constraint function evaluated at the MPP $\mathbf{x}^*$ in $\mathbf{X}$-space. Using the FORM, Eq. (15) can be rewritten as

$$P(G_i(\mathbf{X}) > 0) - \Phi(-\beta_{t_i}) \leq 0 \Rightarrow G_i(\mathbf{x}^*) \leq 0 \tag{16}$$

where $P_{F_i}^{\text{tar}} = \Phi(-\beta_{t_i})$ and β_{t_i} is the target reliability index.

To satisfy the feasibility of the constraint, the MPP needs to be estimated for each constraint by solving the following optimization problem:

$$\text{maximize } g_i(\mathbf{u}) \qquad \text{subject to } \|\mathbf{u}\| = \beta_{t_i} \tag{17}$$

where $g_i(\mathbf{u})$ is the ith constraint function that is transformed from the original space ($\mathbf{X}$-space) into the standard Gaussian space ($\mathbf{U}$-space):

that is, $g_i(\mathbf{u}) \equiv G_i(\mathbf{x}(\mathbf{u})) = G_i(\mathbf{x})$. Although the PMA finds the MPP by maximizing $g_i(\mathbf{u})$ subject to $\|\mathbf{u}\| = \beta_{t_i}$, a reliability index approach does it by minimizing $\|\mathbf{u}\|$ subject to $g_i(\mathbf{u})$ as [16]

$$\text{minimize } \|\mathbf{u}\| \qquad \text{subject to } g_i(\mathbf{u}) = 0 \tag{18}$$

Because the PMA is much more robust in search of the MPP, PMA became the more popular method recently [15]. To further improve stability and efficiency of the PMA for RBDO, an enhanced performance measure approach (PMA+) was developed in [17], which is used in this paper.

The optimum MPP of Eq. (17) is denoted by $\mathbf{u}^*$ in $\mathbf{U}$-space or $\mathbf{x}^*$ in $\mathbf{X}$-space. If the constraint function at the MPP, $g_i(\mathbf{u}^*)$, is less than or equal to zero, then the ith constraint in Eq. (13) is satisfied for the given target reliability. Thus, Eq. (13) can be rewritten as

$$\begin{aligned} &\text{minimize } \text{cost}(\mathbf{d}) \qquad \text{subject to } G_i(\mathbf{x}^*) \le 0, \qquad i = 1, \ldots, \text{nc} \\ &\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \qquad \mathbf{d} \in R^{\text{ndv}}, \qquad \mathbf{X} \in R^n \end{aligned} \tag{19}$$

As shown in Eq. (17), the correlated input variables need to be transformed into the independent standard Gaussian variables using the Rosenblatt transformation, which is defined as the successive conditioning:

$$\begin{aligned} &\Phi(u_1) = F_{X_1}(x_1) \qquad \Phi(u_2) = F_{X_2}(x_2|x_1) \qquad \vdots \\ &\Phi(u_i) = F_{X_i}(x_i|x_1, x_2, \cdots, x_{i-1}) \qquad \vdots \\ &\Phi(u_j) = F_{X_j}(x_j|x_1, x_2, \cdots x_i, \cdots, x_{j-1}) \qquad \vdots \\ &\Phi(u_n) = F_{X_n}(x_n|x_1, x_2, \cdots, x_{n-1}) \end{aligned} \tag{20}$$

If the ordering of the variable x_i is changed into the variable x_j, Eq. (20) can be rewritten as

$$\begin{aligned} &\Phi(u_1) = F_{X_1}(x_1) \qquad \Phi(u_2) = F_{X_2}(x_2|x_1) \qquad \vdots \\ &\Phi(u_i) = F_{X_j}(x_j|x_1, x_2, \cdots, x_{i-1}) \qquad \vdots \\ &\Phi(u_j) = F_{X_i}(x_i|x_1, x_2, \cdots x_{i-1}, x_j, x_{i+1}, \cdots, x_{j-1}) \qquad \vdots \\ &\Phi(u_n) = F_{X_n}(x_n|x_1, x_2, \cdots, x_{n-1}) \end{aligned} \tag{21}$$

Thus, if the number of variables is n, there are $n!$ ways of transforming the original variables into independent standard Gaussian variables. Even though there are many different ways to transform the original variables into independent standard Gaussian variables, because the Rosenblatt transformation is exact, if the inverse reliability analysis in the independent standard Gaussian space is exact, then we should obtain the same results. However, if the FORM is used for the inverse reliability analysis, certain orders of transformation might yield more errors than other orders of transformation, according to the input joint CDF type.

B. Effect of Transformation Ordering for Various Input Joint CDF Types

The joint CDFs of input random variables can be categorized as follows: independent joint CDF, joint CDF modeled by an elliptical copula, and joint CDF modeled by a nonelliptical copula. For various input joint CDF types, to study the effect of transformation ordering in the inverse reliability analysis (PMA), it is necessary to investigate whether the same MPP is obtained for different transformation orderings. However, because the MPPs depend on constraint functions, it is not convenient to compare the MPPs for all constraint functions. Instead of comparing the MPPs, comparing target contours in $\mathbf{X}$-space that are obtained by transforming the target hypersphere $\|\mathbf{u}\| = \beta_{t_i}$ in $\mathbf{U}$-space for different transformation orderings is more appropriate, because obtaining the same target contours means obtaining the same MPPs in $\mathbf{X}$-space, which will lead to the same RBDO result.

First, when the input variables are independent, the transformed target contours from $\mathbf{U}$-space to $\mathbf{X}$-space can be obtained using Eq. (20), which means the Rosenblatt transformation with a given ordering as

$$\begin{aligned} \|\mathbf{u}\|^2 = \mathbf{u}^\mathbf{T}\mathbf{u} &= [\Phi^{-1}(F_{X_1}(x_1))]^2 + \cdots + [\Phi^{-1}(F_{X_i}(x_i))]^2 + \cdots \\ &+ [\Phi^{-1}(F_{X_j}(x_j))]^2 + \cdots + [\Phi^{-1}(F_{X_n}(x_n))]^2 = \beta_{t_i}^2 \end{aligned} \tag{22}$$

When the order of the variable x_i is interchanged into the variable x_j for the second ordering, the transformed target contour is obtained as

$$\begin{aligned} \|\mathbf{u}\|^2 = \mathbf{u}^\mathbf{T}\mathbf{u} &= [\Phi^{-1}(F_{X_1}(x_1))]^2 + \cdots + [\Phi^{-1}(F_{X_j}(x_j))]^2 + \cdots \\ &+ [\Phi^{-1}(F_{X_i}(x_i))]^2 + \cdots + [\Phi^{-1}(F_{X_n}(x_n))]^2 = \beta_{t_i}^2 \end{aligned} \tag{23}$$

which results in the same transformed target contour with Eq. (22). Thus, there is no effect of transformation ordering when the input variables are independent.

Second, consider when the input variables are correlated with a joint CDF modeled by the elliptical copula. In the joint CDF modeled by the elliptical copula, each variable $e_i = \Psi^{-1}[F_{X_i}(x_i)]$ for $i = 1, \ldots, n$ is the standard elliptical variable with the covariance matrix $\mathbf{P}'$, where $\Psi^{-1}(\cdot)$ is the inverse of the elliptical CDF. Because the reduced correlation coefficient ρ'_{ij} between E_i and E_j is different from the correlation coefficient ρ_{ij} between X_i and X_j, it needs to be calculated from ρ_{ij}. The reduced correlation coefficient ρ'_{ij} is obtained from the correlation coefficient ρ_{ij} using the following equation:

$$\rho_{ij} = E[\Xi_i \Xi_j] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \xi_i \xi_j \psi(e_i, e_j; \rho'_{ij}) \mathrm{d}e_i \mathrm{d}e_j \tag{24}$$

where $\Xi_i = (X_i - \mu_{X_i})/\sigma_{X_i}$ is the normalized random variable of X_i, ξ_i is the realization of Ξ_i, and $\psi(\cdot)$ represents the elliptical PDF. The reduced correlation coefficient ρ'_{ij} is obtained by implicitly solving Eq. (24) using a commercial program such as MATLAB.

For the joint CDF modeled by an elliptical copula, because the independent standard Gaussian variables $\mathbf{U}$ correspond to the uniformly distributed variables $\mathbf{S}$ in Eq. (4), the transformation from the correlated standard elliptical variables into independent standard Gaussian variables is linear as

$$\mathbf{u} = \mathbf{L}^{-1}\mathbf{e} \tag{25}$$

where $\mathbf{e}$ represents the realization of the vector $\mathbf{E}$, which is the vector of the correlated standard elliptical variables. If the elliptical copula is Gaussian, $\mathbf{e}$ represents the correlated Gaussian variable, which is the same as $\mathbf{X}$ in Eq. (4). If it is a t copula, $\mathbf{e}$ is a standard t variable $\mathbf{X}$ over a random variable R, defined as $\mathbf{X}/R$ in Eq. (4), where R^2 has an inverse gamma distribution $Ig(\frac{1}{2}\nu, \frac{1}{2}\nu)$. $\mathbf{L}^{-1}$ is the inverse of a lower triangular matrix $\mathbf{L}$ obtained from the Cholesky decomposition of $\mathbf{P}'$. That is, $\mathbf{P}' = \mathbf{L}\mathbf{L}^\mathbf{T}$ and each entry of the matrix $\mathbf{L}$ is obtained as

$$l_{ij} = \begin{cases} \sqrt{(1 - \sum_{k=1}^{i-1} l_{ik}^2)}, & i = j \\ \left(\rho'_{ij} - \sum_{k=1}^{j-1} l_{ik} l_{jk}\right) \Big/ l_{jj}, & i > j \end{cases} \tag{26}$$

Using Eq. (25), the transformed target contour can be expressed as

$$\mathbf{u}^\mathbf{T}\mathbf{u} = \mathbf{e}^\mathbf{T}(\mathbf{L}^{-1})^\mathbf{T}\mathbf{L}^{-1}\mathbf{e} = \mathbf{e}^\mathbf{T}(\mathbf{P}')^{-1}\mathbf{e} \tag{27}$$

where $\mathbf{e} = [e_1, \ldots, e_i, \ldots, e_j, \ldots, e_n]^\mathbf{T}$, and $\mathbf{P}'$ is

$$\mathbf{P}' = \begin{bmatrix} 1 & \rho'_{12} & \cdots & \rho'_{1i} & \cdots & \rho'_{1j} & \cdots & \rho'_{1n} \\ & 1 & & \rho'_{2i} & & \rho'_{2j} & & \rho'_{2n} \\ & & & \vdots & & \vdots & & \vdots \\ & & & 1 & & \rho'_{ij} & & \rho'_{in} \\ & & & & & \vdots & & \vdots \\ & & \text{sym.} & & & 1 & & \rho'_{jn} \\ & & & & & & & \vdots \\ & & & & & & & 1 \end{bmatrix} \tag{28}$$

If the order is changed [i.e., the order of the ith and jth variables are interchanged ($i < j$)], Eq. (25) is changed to

$$\bar{\mathbf{u}} = \mathbf{L}_1^{-1}\mathbf{e}_1 \tag{29}$$

where the vector of the elliptical variable with the interchanged order represents $\mathbf{e}_1 = [e_1, \ldots, e_j, \ldots, e_i, \ldots, e_n]^{\mathbf{T}}$. $\mathbf{L}_1$ is obtained from the Cholesky decomposition of $\mathbf{P}'_1$ (i.e., $\mathbf{P}'_1 = \mathbf{L}_1\mathbf{L}_1^{\mathbf{T}}$), where $\mathbf{P}'_1$ is defined as

$$\mathbf{P}'_1 = \begin{bmatrix} 1 & \rho'_{12} & \cdots & \rho'_{1j} & \cdots & \rho'_{1i} & \cdots & \rho'_{1n} \\ & 1 & & \rho'_{2j} & & \rho'_{2i} & & \rho'_{2n} \\ & & & \vdots & & \vdots & & \vdots \\ & & & 1 & & \rho'_{ji} & & \rho'_{jn} \\ & & & & & \vdots & & \vdots \\ & & \text{sym.} & & & 1 & & \rho'_{in} \\ & & & & & & & \vdots \\ & & & & & & & 1 \end{bmatrix} \tag{30}$$

and

$$\mathbf{L}_1 = \begin{bmatrix} L_{11} & 0 & & & \cdots & & & 0 \\ L_{21} & L_{22} & 0 & & \cdots & & & 0 \\ \vdots & & \ddots & & & & & \vdots \\ L_{i1} & L_{i2} & \cdots & L_{ii} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & & & \vdots \\ L_{j1} & L_{j2} & \cdots & L_{ji} & \cdots & L_{jj} & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{ni} & \cdots & L_{nj} & \cdots & L_{nn} \end{bmatrix} \tag{31}$$

Then the transformed target contour with the interchanged order is given by

$$\bar{\mathbf{u}}^{\mathbf{T}}\bar{\mathbf{u}} = \mathbf{e}_1^{\mathbf{T}}\left(\mathbf{L}_1^{-1}\right)^{\mathbf{T}}\mathbf{L}_1^{-1}\mathbf{e}_1 = \mathbf{e}_1^{\mathbf{T}}\left(\mathbf{P}'_1\right)^{-1}\mathbf{e}_1 \tag{32}$$

To show that the interchanged ordering provides the same transformed target contours [i.e., $\mathbf{u}^{\mathbf{T}}\mathbf{u} = \bar{\mathbf{u}}^{\mathbf{T}}\bar{\mathbf{u}}$ in Eqs. (27) and (32)], Eq. (32) needs to be expressed in terms of $\mathbf{e}$ instead of $\mathbf{e}_1$. For this, another matrix $\mathbf{L}_2$ is introduced instead of $\mathbf{L}_1$. Because the vector of the elliptical variable $\mathbf{e}$ with the original order can be obtained by interchanging the ith row with the jth row of $\mathbf{L}_1$ such that $\mathbf{e} = \mathbf{L}_2\bar{\mathbf{u}}$, the matrix $\mathbf{L}_2$ can be obtained as

$$\mathbf{L}_2 = \begin{bmatrix} L_{11} & 0 & & & \cdots & & & 0 \\ L_{21} & L_{22} & 0 & & \cdots & & & 0 \\ \vdots & \vdots & \ddots & & & & & \vdots \\ L_{j1} & L_{j2} & \cdots & L_{ji} & \cdots & L_{jj} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ L_{i1} & L_{i2} & \cdots & L_{ii} & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{ni} & \cdots & L_{nj} & \cdots & L_{nn} \end{bmatrix} \tag{33}$$

The target contours are given as

$$\bar{\mathbf{u}}^{\mathbf{T}}\bar{\mathbf{u}} = \mathbf{e}_1^{\mathbf{T}}\left(\mathbf{P}'_1\right)^{-1}\mathbf{e}_1 = \mathbf{e}^{\mathbf{T}}\left(\mathbf{L}_2^{-1}\right)^{\mathbf{T}}\mathbf{L}_2^{-1}\mathbf{e} = \mathbf{e}^{\mathbf{T}}\left(\mathbf{P}'_2\right)^{-1}\mathbf{e} \tag{34}$$

To show that $\mathbf{u}^{\mathbf{T}}\mathbf{u} = \bar{\mathbf{u}}^{\mathbf{T}}\bar{\mathbf{u}}$ in Eqs. (27) and (34), it needs to be shown that $\mathbf{P}' = \mathbf{P}'_2$ or $\mathbf{L}\mathbf{L}^{\mathbf{T}} = \mathbf{L}_2\mathbf{L}_2^{\mathbf{T}}$. Consider two arbitrary correlation coefficients of $\mathbf{P}'_1$. For any arbitrary kth column, the entry at the ith row of $\mathbf{P}'_1$, $(\mathbf{P}'_1)_{ik}$, and the one in the jth row of $\mathbf{P}'_1$, $(\mathbf{P}'_1)_{jk}$, are ρ'_{jk} and ρ'_{ik}, respectively, as shown in Eq. (30). Because ρ'_{jk} and ρ'_{ik} are the entries at the jth row and kth column of $\mathbf{P}'$ and at the ith row and kth column of $\mathbf{P}'$, respectively, all entries of $\mathbf{P}'$ are the same as those of $\mathbf{P}'_2$, as follows:

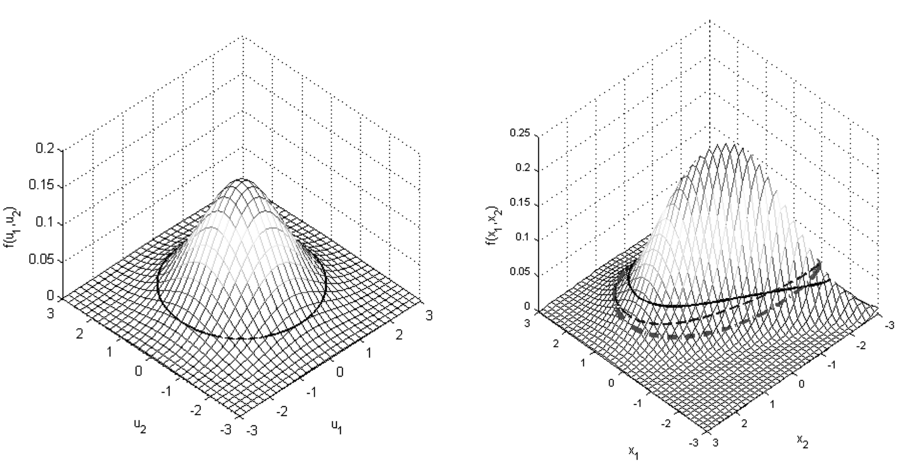
$$\begin{aligned}(\mathbf{P}'_1)_{ik} = \rho'_{jk} = (\mathbf{P}')_{jk} &= i\text{th row of } \mathbf{L}_1 \times k\text{th column of } \mathbf{L}_1^{\mathbf{T}} \\ &= j\text{th row of } \mathbf{L}_2 \times k\text{th column of } \mathbf{L}_2^{\mathbf{T}} = (\mathbf{P}'_2)_{jk}\end{aligned} \tag{35}$$

and

$$\begin{aligned}(\mathbf{P}'_1)_{jk} = \rho'_{ik} = (\mathbf{P}')_{ik} &= j\text{th row of } \mathbf{L}_1 \times k\text{th column of } \mathbf{L}_1^{\mathbf{T}} \\ &= i\text{th row of } \mathbf{L}_2 \times k\text{th column of } \mathbf{L}_2^{\mathbf{T}} = (\mathbf{P}'_2)_{ik}\end{aligned} \tag{36}$$

This means that the transformed target contours are the same, even for different transformation orderings of input variables. Therefore, the transformation is independent of ordering for the joint CDF modeled by an elliptical copula.

Finally, consider when the input variables have a joint CDF modeled by a nonelliptical copula. For example, let two random



a) U-space b) X-space

Fig. 1 Target hypersphere in U-space and target contours for $\beta_t = 2.0$ in X-space.

Table 1 Gaussian quadrature points and weights [6]

N	Quadrature points	Weights
1	0.0	1.0
3	$\pm\sqrt{3}$	0.166667
	0.0	0.666667
	±2.856970	0.011257
5	±1.355626	0.222076
	0.0	0.533333

variables have a joint CDF modeled by the Clayton copula, which is one of the Archimedean copulas, with the marginal Gaussian CDFs X_1 and $X_2 \sim N(0, 1^2)$. The Clayton copula is defined as

$$C(z_1, z_2|\theta) = \left[z_1^{-\theta} + z_2^{-\theta} - 1\right]^{-1/\theta} \quad \text{for } \theta > 0 \tag{37}$$

where the generator is $\varphi_\theta(t) = \frac{1}{\theta}(t^{-\theta} - 1)$, $z_1 = \Phi(x_1)$, $z_2 = \Phi(x_2)$, and θ is the correlation parameter of the Clayton copula. In the Clayton copula, using Kendall's tau τ obtained from samples, θ can be expressed as

$$\theta = \frac{2\tau}{1-\tau} \tag{38}$$

Using the Clayton copula, the Rosenblatt transformation can be carried out in two different ways as

$$\begin{aligned}\Phi(u_1) &= F_{X_1}(x_1) = \Phi(x_1)\\ \Phi(u_2) &= F_{X_2}(x_2|x_1) = \Phi(x_1)^{-\theta-1}[\Phi(x_1)^{-\theta} + \Phi(x_2)^{-\theta} - 1]^{-1/\theta-1}\end{aligned} \tag{39}$$

and

$$\begin{aligned}\Phi(u_1) &= F_{X_2}(x_2) = \Phi(x_2)\\ \Phi(u_2) &= F_{X_1}(x_1|x_2) = \Phi(x_2)^{-\theta-1}[\Phi(x_1)^{-\theta} + \Phi(x_2)^{-\theta} - 1]^{-1/\theta-1}\end{aligned} \tag{40}$$

Using Eqs. (39) and (40), the target hypersphere can be expressed in terms of x_1 and x_2:

$$\mathbf{u}^{\mathbf{T}}\mathbf{u} = u_1^2 + u_2^2 = x_1^2 + (\Phi^{-1}(\Phi(x_1)^{-\theta-1}[\Phi(x_1)^{-\theta} + \Phi(x_2)^{-\theta} - 1]^{-1/\theta-1})) = \beta_t^2 \tag{41}$$

and

$$\mathbf{u}^{\mathbf{T}}\mathbf{u} = u_1^2 + u_2^2 = x_2^2 + (\Phi^{-1}(\Phi(x_2)^{-\theta-1}[\Phi(x_1)^{-\theta} + \Phi(x_2)^{-\theta} - 1]^{-1/\theta-1})) = \beta_t^2 \tag{42}$$

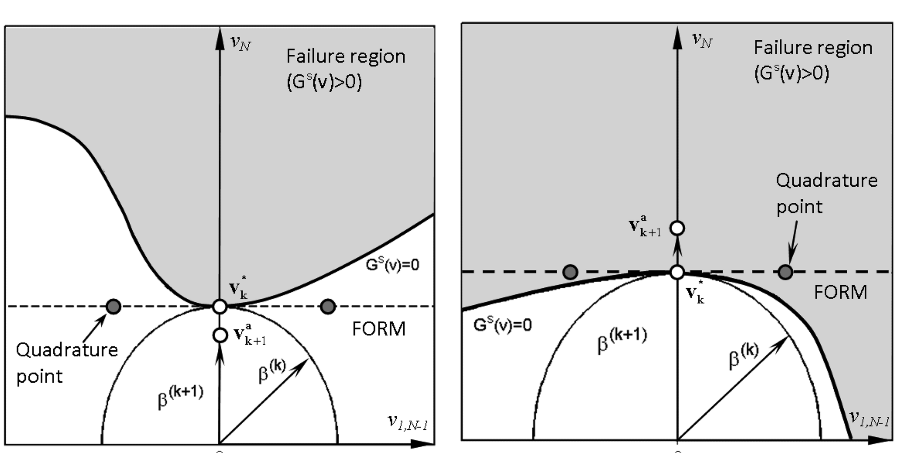
Figure 1a shows the target hypersphere for $\beta_t = 2.0$ in **U**-space with a normalized independent PDF, and Fig. 1b shows two target contours for $\beta_t = 2.0$ transformed from **U**-space to **X**-space, indicated in a dashed-dotted ring (ordering 1) and dashed ring (ordering 2) with a PDF modeled by the Clayton copula. If input variables are independent or correlated with the elliptical copula, the target hypersphere in **U**-space (solid ring in Fig. 1a) is transformed into target contours in **X**-space that are parallel to the X_1–X_2 plane. Therefore, the target contours will be the same in **X**-space for different transformation orderings. However, when input variables are correlated with nonelliptical copula, because of nonlinear Rosenblatt transformation, the target hypersphere in **U**-space is differently transformed to **X**-space according to the transformation ordering, which is not parallel to the X_1–X_2 plane, as shown by the dashed and dashed-dotted rings in Fig. 1b. That is, the contours for the given β_t projected on the X_1–X_2 plane are different from a contour with constant PDF values (solid ring in Fig. 1b), whereas the normalized independent PDF contour for a certain PDF value and the target hypersphere will coincide. However, note that the inverse reliability analysis is carried out in **U**-space, not **X**-space.

It might be possible that MPPs obtained from different transformation orderings could be the same even though the contours for β_t in **X**-space are different. However, because the MPPs depend on both the contour shapes and constraint functions, it is not easy to predict how the MPPs depend on the contour shape. As seen in Fig. 1b, the two contours for β_t have intersection points, where x_1 and x_2 values are the same. If both MPPs happen to be at the intersection point, then even if the contours for β_t in **X**-space are different, these MPPs will be the same. However, it will be extremely rare that both MPPs are at the same intersection point of two differently transformed contours.

Thus, for the nonelliptical joint CDF modeled by a nonelliptical copula, the Rosenblatt transformation becomes highly nonlinear, which cannot be handled accurately by the FORM. A more accurate method than the FORM for the estimation of the probability of failure in the reliability analysis is necessary to reduce the effect of the transformation ordering on the RBDO results.

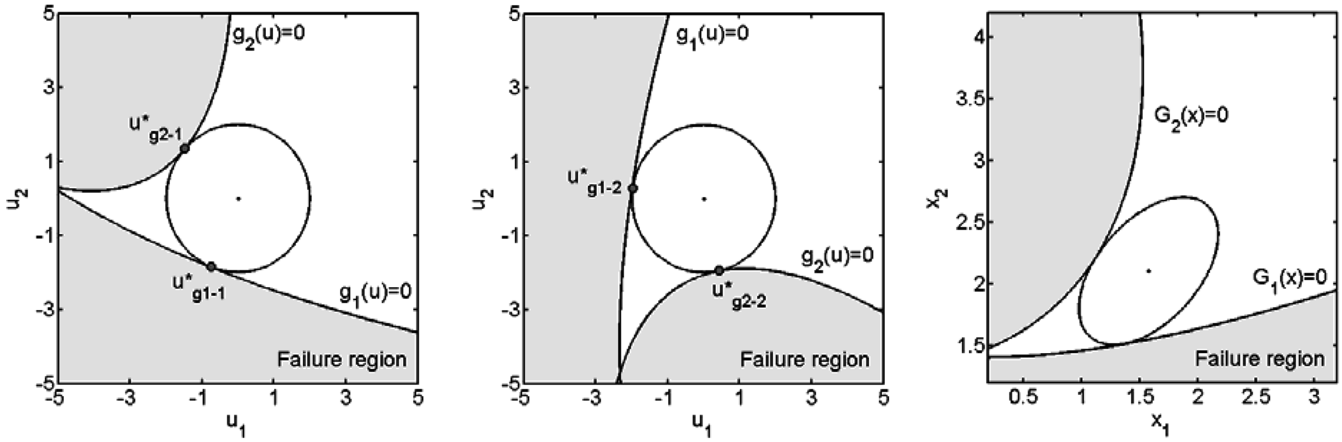
IV. Method to Resolve Effect of Transformation Ordering Using MPP-Based DRM

The MPP-based reliability analyses such as the FORM [15,18,19] and the SORM [20,21] have been very commonly used for reliability assessment. However, when the constraint function is nonlinear or multidimensional, the reliability analysis using the FORM could be erroneous, because the FORM cannot handle the complexity of



a) Concave function b) Convex function

Fig. 2 DRM-based MPP for concave and convex functions [6].



a) Ordering 1 in U-space b) Ordering 2 in U-space c) Two ordering in X-space

Fig. 3 Target hyperspheres for $\beta_t = 2.0$ with different orderings in U-space and target contour for $\beta_t = 2.0$ in X-space using a Gaussian copula.

nonlinear or multidimensional functions. Reliability analysis using the SORM may be accurate, but the second-order derivatives required for the SORM are very difficult and expensive to obtain in industrial applications. On the other hand, the MPP-based DRM achieves both the efficiency of the FORM and the accuracy of the SORM [6].

The DRM is developed to accurately and efficiently approximate a multidimensional integral. There are several DRMs depending on the level of dimension reduction: univariate dimension reduction, bivariate dimension reduction, and multivariate dimension reduction. The univariate, bivariate, and multivariate dimension reductions indicate an additive decomposition of n-dimensional performance function into one, two, and s-dimensional functions ($s \leq n$), respectively. In this paper, the univariate DRM is used for calculating probability of failure, due to its simplicity and efficiency.

The univariate DRM is carried out by decomposing an n-dimensional constraint function $G(\mathbf{X})$ into the sum of one-dimensional functions at the MPP as [6,22,23]

$$G(\mathbf{X}) \cong \hat{G}(\mathbf{X}) \equiv \sum_{i=1}^{n} G(x_1^*, \ldots, x_{i-1}^*, X_i, x_{i+1}^*, \ldots, x_n^*) - (n-1)G(\mathbf{x}^*) \tag{43}$$

where $\mathbf{x}^* = \{x_1^*, x_2^*, \ldots, x_n^*\}^T$ is the FORM-based MPP obtained from Eq. (17), and n is the number of random variables. In the inverse reliability analysis, because the probability of failure cannot be directly calculated in **U**-space, a constraint shift in a rotated standard Gaussian space (**V**-space) needs to be defined as

$$\tilde{G}^s(\mathbf{v}) \equiv \tilde{G}(\mathbf{v}) - \tilde{G}(\mathbf{v}^*) \tag{44}$$

where $\mathbf{v}^* = \{0, \ldots, 0, \beta\}^T$ is the MPP in **V**-space and $\tilde{G}(\mathbf{v}) \equiv G(\mathbf{x}(\mathbf{v}))$. Then, using the shifted constraint function, the probability of failure using the MPP-based DRM is calculated as [6]

$$P_F^{\mathrm{DRM}} = \left[\prod_{i=1}^{n-1} \int_{-\infty}^{\infty} \Phi\left(-\beta + \frac{\tilde{G}_i^s(v_i)}{b_1}\right)\phi(v_i)\mathrm{d}v_i\right] \Big/ \Phi(-\beta)^{n-2} \tag{45}$$

where $\tilde{G}_i^s(v_i) \equiv \tilde{G}^s(0, \ldots, 0, v_i, 0, \ldots, \beta)$ is a function of v_i only and

$$b_1 = \left\| \frac{\partial g(\mathbf{u}^*)}{\partial \mathbf{u}} \right\|$$

Equation (45) can be approximated as using the moment-based integration rule [24] similar to Gaussian quadrature [25]:

$$P_F^{\mathrm{DRM}} = \left[\prod_{i=1}^{n-1} \sum_{j=1}^{N} w_j \Phi\left(-\beta + \frac{\tilde{G}_i^s(v_i^j)}{b_1}\right)\right] \Big/ \Phi(-\beta)^{n-2} \tag{46}$$

where v_i^j represents the jth quadrature point for v_i, w_j denote weights, and N is the number of quadrature points. The quadrature points and weights for the standard Gaussian random variables v_i are shown in Table 1. When the quadrature point is 1 ($N = 1$) and weight is 1, Eq. (46) becomes

$$P_F^{\mathrm{DRM}} \cong \left\{\left[w_1 \prod_{i=1}^{n-1} \Phi\left(-\beta + \frac{\tilde{G}_i^s(v_i^1)}{b_1}\right)\right] \Big/ \Phi(-\beta)^{n-2}\right\} = \left[\prod_{i=1}^{n-1} \Phi(-\beta)\right] \Big/ \Phi(-\beta)^{n-2} = \Phi(-\beta) \tag{47}$$

where $w_1 = 1$ and $v_i^1 = 0$ by Table 1 and $\tilde{G}_i^s(v_i^1) = \tilde{G}_i^s(0) = 0$. Equation (47) is the same as the probability of failure calculated by the FORM. Therefore, it can be said that the probability of failure calculated by the FORM is a special case of the one calculated by the DRM with one quadrature point and weight.

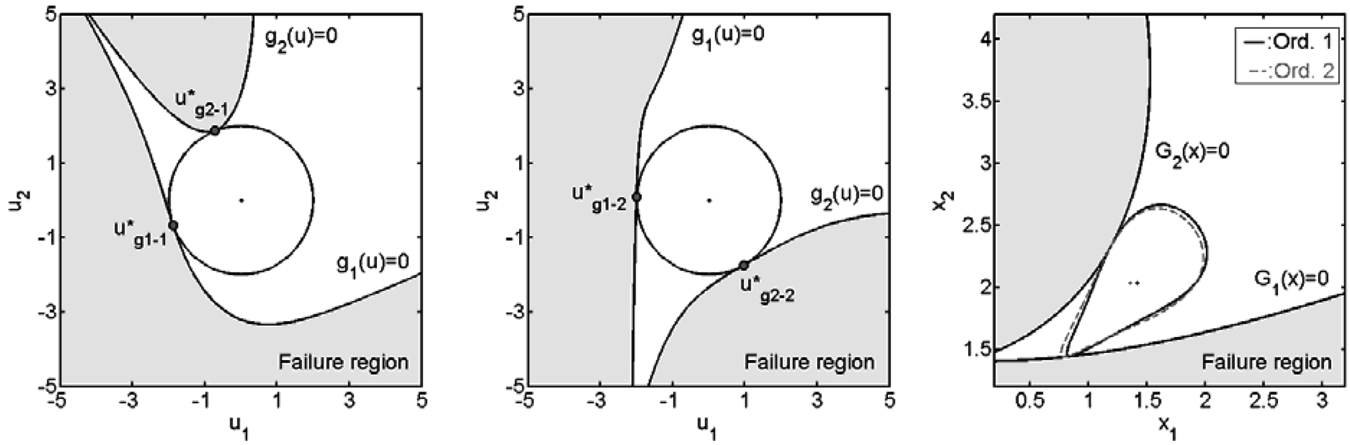
Note that the DRM cannot carry out the inverse reliability analysis directly. Using the probability of failure obtained using the DRM at the FORM-based MPP, the initial β_t can be updated. Then a new inverse reliability analysis is carried out using Eq. (17) to find a better MPP based on the updated β. By doing this, the DRM helps us to find a more accurate MPP for multidimensional and highly nonlinear constraint functions. Thus, using the estimated P_F^{DRM} obtained from the MPP-based DRM for the shifted constraint function $G^s(\mathbf{x})$, the corresponding reliability index β_{DRM} can be defined as

$$\beta_{\mathrm{DRM}} = -\Phi^{-1}\left(P_F^{\mathrm{DRM}}\right) \tag{48}$$

which is not the same as the target reliability index $\beta_t = -\Phi^{-1}\left(P_F^{\mathrm{Tar}}\right)$, because the nonlinearity of the constraint function is

Table 2 RBDO results obtained from a Gaussian copula ($P_f^{\mathrm{tar}} = 2.275\%$)

Case	Cost	Optimum design points	G_1	G_2	G_3	P_{f1}, %	P_{f2}, %
FORM	3.678	1.574, 2.104	0.000	0.000	−1.790	2.342	1.722
DRM3	3.653	1.548, 2.105	0.000	0.000	−1.805	2.208	2.142
DRM5	3.651	1.546, 2.105	0.000	0.000	−1.807	2.256	2.276



a) Ordering 1 in U-space b) Ordering 2 in U-space c) Two ordering in X-space

Fig. 4 Target hyperspheres for $\beta_t = 2.0$ with different orderings in U-space and target contour for $\beta_t = 2.0$ in X-space using a Clayton copula.

reflected in the calculation of P_F^{DRM}. Hence, using β_{DRM}, a new updated reliability index β_{up} can be defined as

$$\beta_{\mathrm{up}} \equiv \beta_{\mathrm{cur}} + \Delta\beta = \beta_{\mathrm{cur}} + (\beta_t - \beta_{\mathrm{DRM}}) \tag{49}$$

where β_{cur} is the current reliability index. The recursive form of the Eq. (49) is

$$\beta^{(k+1)} \equiv \beta^{(k)} + \Delta\beta = \beta^{(k)} + (\beta_t - \beta_{\mathrm{DRM}}) \tag{50}$$

where $\beta^{(0)} = \beta_t$ at the initial step.

Using this updated reliability index, the updated MPP can be found by using an iterative MPP search or using an approximation. If an iterative MPP search with the updated reliability index is used, the updated MPP is called the true DRM-based MPP and is denoted by $\mathbf{x}^*_{\mathrm{DRM}}$, which means that the updated MPP is the optimum solution of Eq. (17) using β_{up} instead of β_t; however, the procedure to find the new MPP search for every updated reliability index will be computationally expensive. Accordingly, to improve the efficiency of the optimization, the updated MPP can be approximated as [26]

$$\mathbf{u}^a_{k+1} \cong \frac{\beta^{(k+1)}}{\beta^{(k)}}\mathbf{u}^*_k \quad \text{or} \quad \mathbf{v}^a_{k+1} \cong \frac{\beta^{(k+1)}}{\beta^{(k)}}\mathbf{v}^*_k \tag{51}$$

assuming that the updated MPP $\mathbf{v}^a_{k+1}$ is located along the same radial direction v_N as the current MPP $\mathbf{v}^*_k$ in **V**-space, as shown in Fig. 2. The detailed explanation on this procedure is presented in [6]. The updated MPP obtained from Eq. (51) is called the DRM-based MPP and is used to check whether or not the optimum design satisfies the constraint. The location of the DRM-based MPP for a concave and convex function is shown in Figs. 2a and 2b, respectively.

Similar to the FORM, using the DRM-based inverse reliability analysis, the RBDO formulation in Eq. (13) can be rewritten as

$$\text{minimize cost}(\mathbf{d}) \qquad \text{subject to } G_i(\mathbf{x}^*_{\mathrm{DRM}}) \le 0$$
$$i = 1, \ldots, \mathrm{nc}, \qquad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \qquad \mathbf{d} \in R^{\mathrm{ndv}}, \qquad \mathbf{X} \in R^n \tag{52}$$

where $\mathbf{x}^*_{\mathrm{DRM}}$ is the MPP obtained from the DRM and $\mathbf{x}^*$ is the one obtained from the FORM.

V. Numerical Examples

To observe how the ordering of transformation affects the RBDO results for the joint CDFs modeled by elliptical copula and nonelliptical copulas, two- and four-dimensional mathematical problems are tested.

A. Two-Dimensional Problem

Suppose that two input variables are correlated with the Gaussian copula, where Pearson's rho is given as $\rho = 0.5$. The RBDO formulation is defined as

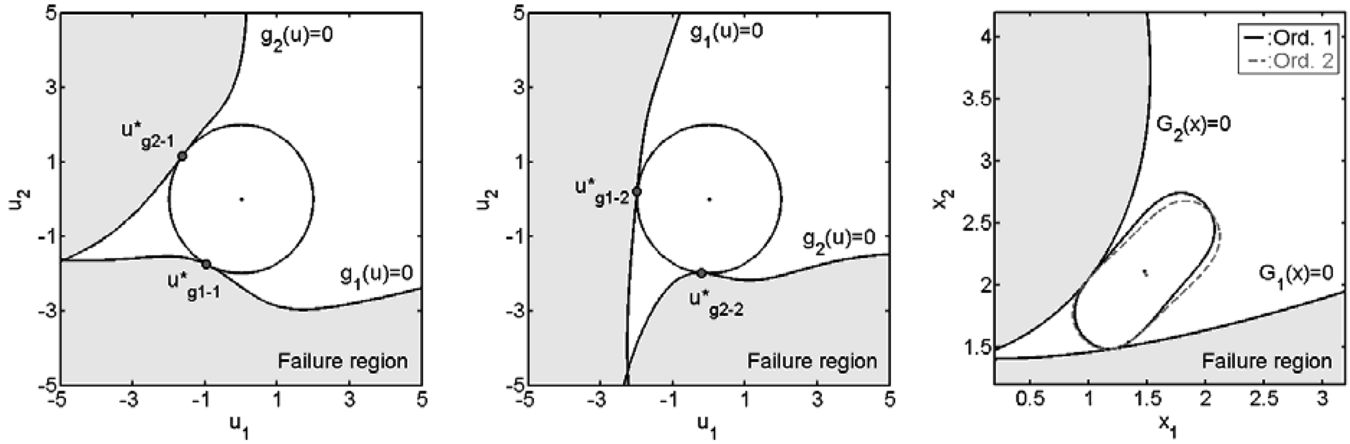
$$\begin{aligned}
&\text{minimize } \text{cost}(\mathbf{d}) = d_1 + d_2\\
&\text{subject to } P(G_i(\mathbf{X}) \ge 0) \le P_F^{\mathrm{tar}}, \qquad i = 1,2,3\\
&0 \le d_1, d_2 \le 10, \qquad P_F^{\mathrm{tar}} = 2.275\%\\
&G_1(\mathbf{X}) = 1 - (0.4339X_1 - 0.9010X_2 - 1.5)^2(0.9010X_1\\
&\quad + 0.4339X_2 + 2)/20\\
&G_2(\mathbf{X}) = 1 - (X_1 + X_2 - 2.8)^2/30 - (X_1 - X_2 + 12)^2/120\\
&G_3(\mathbf{X}) = 1 - 80/(8X_1 + X_2^2 + 5)
\end{aligned} \tag{53}$$

where the marginal CDFs are Gaussian, given by X_1 and $X_2 \sim N(5.0, 0.3^2)$. Denote the initial ordering as ordering 1 and the interchanged ordering ($x_1 \leftrightarrow x_2$) as ordering 2.

When input variables are correlated with the Gaussian copula (even if the MPPs are different in **U**-space for different orderings, as shown in Figs. 3a and 3b), the target contours are the same, which means that the MPPs are the same in **X**-space, as shown in Fig. 3c. Thus, the same optimum design points are obtained even for different orderings. Even though there is no effect of transformation ordering, because the second constraint function is nonlinear, the FORM has some error in estimating the probability of failure for the second constraint (P_{f2}). Using FORM and DRM with three and five quadrature points (FORM, DRM3, and DRM5, respectively, in Table 2) RBDO results including the cost, optimum design results, and constraint-function values G_1, G_2, and G_3 evaluated at the MPPs for each constraint are obtained. The probability of failures P_{f1} and P_{f2} for active constraints G_1 and G_2 are calculated at the obtained

Table 3 RBDO results obtained from a Clayton copula ($P_f^{\mathrm{tar}} = 2.275\%$)

Case	Cost	Optimum design points	$\lvert d_1^{\mathrm{opt}} - d_2^{\mathrm{opt}}\rvert$	G_1	G_2	G_3	P_{f1}, %	P_{f2}, %
FORM-1	3.446	1.413, 2.032	0.062	0.000	0.000	−1.606	2.531	1.031
FORM-2	3.386	1.352, 2.034		0.000	0.000	−1.632	2.354	2.108
DRM3-1	3.417	1.380, 2.037	0.034	0.000	0.000	−1.621	2.352	1.582
DRM3-2	3.383	1.347, 2.036		0.000	0.000	−1.633	2.280	2.257
DRM5-1	3.400	1.364, 2.036	0.016	0.000	0.000	−1.630	2.320	1.881
DRM5-2	3.385	1.348, 2.037		0.000	0.000	−1.633	2.276	2.266



a) Ordering 1 in U-space b) Ordering 2 in U-space c) Two orderings in X-space

Fig. 5 Target hyperspheres for $\beta_t = 2.0$ with different orderings in U-space and target contour for $\beta_t = 2.0$ in X-space using a Frank copula.

optimum design points using Monte Carlo simulation. As shown in the second row of Table 2, P_{f2} is poorly estimated (less than the target probability of failure, 2.275%) comparing with P_{f1}. When the MPP-based DRM with three quadrature points is used, the probabilities of failure for both constraint functions (P_{f1} and P_{f2}) are closer to the target probability of failure. When the MPP-based DRM with five quadrature points is used, P_{f1} and P_{f2} are almost the same as the target probability of failure. Thus, the FORM error can be reduced using the MPP-based DRM.

Suppose that two input variables are correlated by a Clayton copula with Kendall's tau $\tau = 0.5$. The same RBDO problem in Eq. (53) is tested.

As shown in Figs. 4a and 4b, the target hyperspheres in **U**-space are the same, but the constraint functions in **X**-space are differently transformed in **U**-space according to the different transformation orderings. Because the transformation of the nonelliptical copula is highly nonlinear, some transformed constraint functions become highly nonlinear in **U**-space. For ordering 1 (Fig. 4a), the first constraint function is mildly nonlinear near the MPP (u^*_{g1-1}), but the second constraint function is highly nonlinear near the corresponding MPP (u^*_{g2-1}), which yields a large FORM error. On the other hand, for ordering 2 (Fig. 4b), two constraint functions are mildly nonlinear near the MPPs (u^*_{g1-2} and u^*_{g2-2}), so that the FORM estimates the probability of failure more accurately than that of the second constraint function with the first ordering. The target hyperspheres in Figs. 4a and 4b can be transformed as two different target contours for $\beta_t = 2.0$ (Fig. 4c) in **X**-space according to different orderings. These different contours for $\beta_t = 2.0$ provide different optimum design points, as shown in Fig. 4c and in the FORM results of Table 3.

In Table 3, case FORM-1 and case FORM-2 indicate the FORM with orderings 1 and 2, respectively. As expected, when the FORM is used for the ordering 1 (FORM-1), the probability of failure for the second constraint (P_{f2}) is poorly estimated (much less than target probability of 2.275%). As a result, the optimum design points obtained using the FORM with different orderings are indeed different, as shown in the third and fourth columns of Table 3. If the MPP-based DRM with three quadrature points, denoted as DRM3-1 and DRM3-2 for orderings 1 and 2, respectively, is used, the difference between optimum design results is reduced from 0.062 to 0.034 and the DRM provides a more accurate estimation of the probabilities of failure (closer to 2.275%) for both orderings. If the number of quadrature points is five (DRM5-1 and DRM5-2), then the optimum design points are much closer to each other for both orderings and the probability of failure calculation also becomes more accurate. To estimate the probability of failures more accurately for highly nonlinear constraint functions such as the second constraint shown in Fig. 4a, more than five quadrature points might be necessary. However, increasing the number of quadrature points means increasing computational effort. Therefore, three or five quadrature points for the MPP-based DRM are usually used.

For the same problem in Eq. (53), assume that two input variables are now correlated with a Frank copula, which belongs to the Archimedean copula, with Kendall's tau $\tau = 0.5$. The Frank copula is given as

$$C(z_1, z_2|\theta) = -\frac{1}{\theta}\ln\left(1 + \frac{(e^{-\theta z_1} - 1)(e^{-\theta z_2} - 1)}{e^{-\theta} - 1}\right) \tag{54}$$

where $z_1 = \Phi(x_1)$ and $z_2 = \Phi(x_2)$ with X_1 and $X_2 \sim N(5.0, 0.3^2)$. The correlation parameter θ can be calculated from Kendall's tau by solving the following equation:

$$\tau = 1 - \frac{4}{\theta}\left(1 - \frac{1}{\theta}\int_0^t \frac{\theta}{e^t - 1}\,\mathrm{d}t\right) \tag{55}$$

As observed in the previous example, when the joint CDF modeled by a nonelliptical copula is used, the MPPs obtained from differently transformed constraint functions provide different RBDO results according to the different ordering of input variables.

Because of the nonlinear transformation of the joint CDF modeled by a nonelliptical copula, the constraint functions in **X**-space are differently transformed into those in **U**-space for different orderings, and some transformed constraint functions are highly nonlinear (Figs. 5a and 5b). If the target hypersphere is transformed from **U**-space to **X**-space, it becomes two different target contours for $\beta_t = 2.0$ in **X**-space for different transformation orderings, which result in different optimum design points (Fig. 5c). For ordering 1,

Table 4 RBDO results obtained from a Frank copula ($P_f^{\text{tar}} = 2.275\%$)

Case	Cost	Optimum design points	$\lvert d_1^{\text{opt}} - d_2^{\text{opt}}\rvert$	G_1	G_2	G_3	P_{f1}, %	P_{f2}, %
FORM-1	3.590	1.477, 2.114	0.036	0.000	0.000	−1.481	1.765	2.287
FORM-2	3.572	1.491, 2.081		0.000	0.000	−1.451	2.351	1.638
DRM3-1	3.541	1.455, 2.086	0.015	0.000	0.000	−1.503	2.171	2.255
DRM3-2	3.551	1.469, 2.081		0.000	0.000	−1.461	2.323	1.953
DRM5-1	3.535	1.453, 2.082	0.006	0.000	0.000	−1.506	2.242	2.270
DRM5-2	3.539	1.459, 2.080		0.000	0.000	−1.466	2.285	2.093

Table 5 RBDO results of Rosen–Suzuki problem obtained from Gumbel and A12 copulas

Case	Cost	Optimum design points	G_1	G_2	G_3	P_{f2}, %	P_{f3}, %
FORM-1	−0.144	4.620, 5.623, 6.618, 4.024	−0.561	0.000	0.000	9.635	4.242
FORM-2	−0.144	4.644, 5.591, 6.618, 4.020	−0.561	0.000	0.000	8.192	4.181
FORM-3	−0.145	4.634, 5.561, 6.639, 4.065	−0.561	0.000	0.000	7.110	4.022
FORM-4	−0.145	4.644, 5.584, 6.646, 4.072	−0.561	0.000	0.000	6.298	4.072
DRM3-1	−0.139	4.636, 5.586, 6.574, 4.119	−0.563	0.000	0.000	3.000	2.605
DRM3-2	−0.139	4.636, 5.561, 6.582, 4.123	−0.563	0.000	0.000	2.701	2.612
DRM3-3	−0.139	4.611, 5.564, 6.592, 4.131	−0.562	0.000	0.000	3.109	2.472
DRM3-4	−0.139	4.619, 5.545, 6.590, 4.128	−0.562	0.000	0.000	2.790	2.252
DRM5-1	−0.139	4.637, 5.584, 6.573, 4.124	−0.563	0.000	0.000	2.903	2.605
DRM5-2	−0.139	4.640, 5.558, 6.582, 4.125	−0.563	0.000	0.000	2.617	2.637
DRM5-3	−0.139	4.610, 5.554, 6.593, 4.149	−0.563	0.000	0.000	2.468	2.306
DRM5-4	−0.138	4.612, 5.544, 6.591, 4.149	−0.563	0.000	0.000	2.359	2.133

the first constraint function is highly nonlinear near the MPP (u^*_{g1-1}) (Fig. 5a), whereas for ordering 2, the second constraint function is highly nonlinear near the MPP (u^*_{g2-2}) (Fig. 5b). Therefore, for ordering 1, the FORM error is large for the first constraint (FORM-1), whereas for the second ordering, it is large for the second constraint (FORM-2). As shown in Table 4, probabilities of failure P_{f2} for ordering 1 and P_{f1} for ordering 2 are close to the target probability (2.275%), whereas P_{f1} for ordering 1 and P_{f2} for ordering 2 are not. When the MPP-based DRM with three quadrature points is used, the probability of failure becomes closer to the target probability for both orderings (DRM3-1 and DRM3-2). The DRM with five quadrature points provides the most accurate calculation of the probability of failure (DRM5-1 and DRM5-3). The optimum design points obtained from the DRM are indeed similar to each other, compared with those obtained from the FORM for different orderings. Thus, the DRM is necessary to reduce the effect of transformation ordering and to provide accurate RBDO results. If the number of correlated variables is larger than 2, the effect of transformation ordering and inaccurate estimation of probability of failure might be more significant, and thus using the FORM in the reliability analysis might be more unreliable. In the next section, this issue will be further addressed through a four-dimensional problem.

B. Four-Dimensional Problem

This example is the four-dimensional modified Rosen–Suzuki problem [27] and the RBDO is formulated to

$$\begin{aligned}
&\text{minimize cost}(\mathbf{d}) = \frac{d_1(d_1-15)+d_2(d_2-15)+d_3(2d_3-41)+d_4(d_4-3)+245}{245}\\
&\text{subject to } P(G_i(\mathbf{X})\geq 0)\leq P_F^{\text{Tar}}, \quad i=1,2,3\\
&0\leq d_1,d_2,d_3,d_4\leq 10, \quad P_F^{\text{Tar}}=2.275\%\\
&G_1(\mathbf{X})=1-\frac{X_1(9-X_1)+X_2(11-X_2)+X_3(11-X_3)+X_4(11-X_4)}{68}\\
&G_2(\mathbf{X})=1-\frac{X_1(11-X_1)+2X_2(10-X_2)+X_3(10-X_3)+X_4(21-2X_4)}{151}\\
&G_3(\mathbf{X})=1-\frac{2X_1(9-X_1)+X_2(11-X_2)+X_3(10-X_3)+X_4}{95}
\end{aligned} \tag{56}$$

Assume that the first and second variables are correlated with the Gumbel copula and the third and fourth variables are correlated with the A12 copula, in which the Gumbel and A12 copula belong to the Archimedean copula. The Gumbel copula is defined as

$$C(z_1,z_2|\theta)=\exp\{-[(-\ln z_1)^\theta+(-\ln z_2)^\theta]^{1/\theta}\} \tag{57}$$

where $z_1=\Phi(x_1)$ and $z_2=\Phi(x_2)$ with X_1 and $X_2\sim N(5.0,0.3^2)$. Kendall's tau $\tau=0.5$ is assumed for both copulas and the correlation parameter is obtained as $\theta=1/(1-\tau)$. The A12 copula is defined as

$$C(z_3,z_4|\theta)=\{1+[(z_3^{-1}-1)^\theta+(z_4^{-1}-1)^\theta]^{1/\theta}\}^{-1} \tag{58}$$

Likewise, $z_3=\Phi(x_3)$, $z_4=\Phi(x_4)$ with $X_3,X_4\sim N(5.0,0.3^2)$, and $\theta=2/3(1-\tau)$.

Because the number of input variables is four and two pairs of variables are correlated, four different orderings are possible in the transformation. In Table 5, FORM-1 indicates FORM with the initial ordering, which means that the ordering is not changed. FORM-2 is the case with the interchanged ordering of x_1 and x_2, and FORM-3 is the one with interchanged ordering of x_3 and x_4. FORM-4 is the case in which the orderings of all variables are interchanged, which means that x_1 and x_2 are interchanged and x_3 and x_4 are interchanged. As seen in Table 5, for all orderings, the probabilities of failure P_{f2} and P_{f3} are poorly estimated when the FORM is used. Even though the calculation of probability of failure for the fourth ordering is the most accurate, P_{f2} and P_{f3} are still much larger than the target probability of 2.275%. Compared with the two-dimensional example, the FORM error for the four-dimensional case is more significant. When the MPP-based DRM with three quadrature points is used (DRM3-1, 2, 3, and 4), the difference between the probabilities of failure becomes smaller than when the FORM is used. When five quadrature points are used (DRM5-1, 2, 3, and 4), the MPP-based DRM estimates the probabilities of the failure more accurately than the case with three quadrature points. Thus, the MPP-based DRM is necessary to reduce the ordering effect on RBDO results.

Method 1 is ordering 1 (original ordering). Method 2 is ordering 2 ($X_1 \leftrightarrow X_2$). Method 3 is ordering 3 ($X_3 \leftrightarrow X_4$). Method 4 is ordering 4 ($X_1 \leftrightarrow X_2$ and $X_3 \leftrightarrow X_4$).

VI. Conclusions

In RBDO problems, the joint CDF needs to be used in the Rosenblatt transformation for the inverse reliability analysis. However, because the joint CDFs are difficult to obtain, copulas are proposed to model the joint CDFs in this paper. Incorporating the copula concept, the joint CDF can be categorized as independent joint CDF, joint CDF modeled by the elliptical copula, and joint CDF modeled by the nonelliptical copula. When the input variables are independent or correlated with the elliptical copula, the inverse analysis results are the same for different transformation orderings of the input variables. However, when the correlated input variables with joint CDFs are modeled by the nonelliptical copula, different transformation orderings could lead to highly nonlinear constraint functions. Thus, it becomes a significant challenge to accurately carry out the inverse reliability analysis using the FORM. Thus, the MPP-based DRM, which can handle the nonlinear constraints, is proposed to be used in this paper for the RBDO of problems with correlated input variables with joint CDFs modeled by nonelliptical copulas. Numerical examples show that when the MPP-based DRM is used, the difference between the RBDO results using different transformation orderings is reduced and the accurate estimation of probability of failure is achieved.

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A. Messac
Associate Editor